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TRANSMISSION LINE CALCULATIONS

BY

JOHN O. TAYLOR AND BURTON FRENCH

THESIS

FOR THE DEGREE OF BACHELOR OF SCIENCE IN ELECTRICAL ENGINEERING

IN THE

COLLEGE OF ENGINEERING OF THE UNIVERSITY OF ILLINOIS.

PRESENTED JUNE, 1904.

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May 27, 1904 190

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

BURTON FRENCH and JOHN ORLO TAYLOR

ENTITLED TRANSMISSION LINE CALCULATIONS.

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Bachelor of Science in Electrical Engineering.

Morgan Brooks.

HEAD OF DEPARTMENT OF Electrical Engineering.

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INTRODUCTION

It is the object of this thesis to develop the methods of making all nec-
-essary calculations necessary in consider-
-ing alternating currents on transmis-
-sion lines. Formulae for the capacity and inductance of transmission lines, for finding generator voltage and line drop for transmission line of given resistance and inductance, supplying a given volt-
-age and current at a receiver circuit, for power delivered over line of given resist-
-ance and inductance by a generator of given voltage, are derived. Value of current for which power delivered to re-
-ceiver circuit of given phase angle, over a line of given constants, by a generator

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of given voltage, is a maximum, and value of phase angle of receiver circuit for which current in receiver circuit is a minimum, power in receiver circuit, line constants and generator voltage being given, are found. Approximate and exact formulae for the voltage and current at any point of a transmission line of distributed capacity and inductance are developed.

CURRENT CONSIDERED AS THE CAUSE RATHER THAN THE EFFECT OF ELECTROMOTIVEFORCE.

E. M. F. is usually and, probably properly, especially when a whole circuit is considered, taken as the cause of current; the formula for current being

$$I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}},$$

where R is resistance; L is inductance; C, capacity in series; and E is eletromotiveforce. Electromotiveforce to cause current I to flow through resistance R, equals IR in phase with I, to cause I to flow through inductance L, equals $\omega L I$, $\frac{\pi}{2}$ ahead of I and to cause I to flow into and out of C, equals $\frac{I}{\omega C}$, $\frac{\pi}{2}$ behind I.

In considering a transmission line it is more convenient to consider current the cause and e. m. f. the effect; the

circuit in this case being divided into the two parts, line and receiver. Consider-ing current as the cause of e. m. f., $IR = E$, in phase with I, is the e. m. f. caused by current I flowing through resistance R, $I\omega L$ = e. m. f., $\frac{\pi}{2}$ ahead of I caused by current I flowing through inductance L, and $\frac{I}{\omega C}$ = e. m. f. $\frac{\pi}{2}$ behind I, caused by current I, flowing through C.

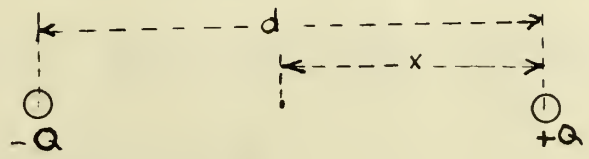
If the voltage, current and phase angle of the receiver circuit and the constants R and x of line, where $x = (\omega L - \frac{1}{\omega C})$, are known, the generator voltage can be found by adding to receiver voltage, the voltage caus-ed by the receiver current flowing in the line. The total e. m. f. in any case is equal to $I\sqrt{R^2 + X^2}$. If the e. m. f. remain the same and $\sqrt{R^2 + X^2}$ increase

I becomes smaller. In the case of open circuit $\sqrt{R^2+X^2} = \infty$ and $I = 0$.

CALCULATION OF CAPACITY.

In calculating the capacity of a line the conductors are assumed charged with equal and opposite quantities of electricity and the potential difference between the conductors calculated. The capacity is the ratio of the quantity on one of the conductors divided by the potential difference between them.

To calculate the capacity between two conductors of distance d from each other and of radius r:



Assume conductors charged with

$+Q$ and $-Q$ units respectively, per unit length. Then since force exerted by a charge on a conductor of infinite length varies inversely as the distance from the conductor the force exerted on a unit conductor is

$$\frac{Q}{x} + \frac{Q}{d-x}$$

Work done in moving unit charge from one conductor to the other is

$$\int_r^{d-r} \left(\frac{Q}{x} + \frac{Q}{d-x}\right) dx$$

$$= \left[Q \log x - Q \log(d-x)\right]_{x=r}^{x=d-r}$$

$$= Q\log(d-r) - Q\log r - Q\log r + Q\log(d-r).$$

$$= 2\,Q \log\left(\frac{d-r}{r}\right)$$

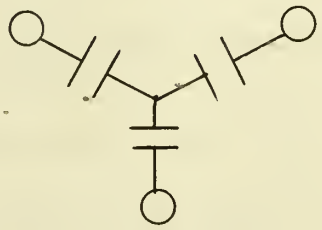
or since for overhead lines r is very small compared with d the potential difference between the lines is

$$E = 2Q\log\frac{d}{r}$$

since $Q = CE$ $\qquad E = 2CE\log\frac{d}{r}, \therefore C = \frac{1}{2\log\left(\frac{d}{r}\right)}$

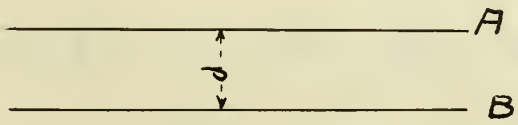
The capacity between one wire and a line or plane of zero potential midway between the wires is $\frac{1}{\log(\frac{d}{r})}$.

In the case of a three phase line it is found by experiment that the capacity is represented by condensers connected in star between the lines instead of by condensers connected delta between the wires.



CALCULATION OF INDUCTANCE.

The inductance of two parallel wires may be determined as follows:-



Let wires be A and A' : r = radius of each wire

I = current flowing

d = distance between wires

μ and μ' = permeability of surrounding mediums

The strength of the magnetic field H_a at a distance, a, from center of A, due to current in A is $H_a = \frac{2I}{a}$. The magnetic induction at that point is $B_a = \frac{2\mu I}{a}$.

Now consider a space cut by two planes perpendicular to the axes of the conductors and of unit distance apart. Within this space at a distance, a, from A, a number of lines of force,

$$B_a da = dN_a = \frac{2\mu I da}{a}$$

will pass through a radical width da

$$\therefore \quad N_a = \int_r^d \frac{2\mu I da}{a} = 2\mu I \log_e \frac{d}{r}$$

is the total number of lines of force passing between the planes, from surface A and centre of A'.

At a point p within A and at a distance b from center of A, the magnetic effect will be as though the current within a circle of radius b, were condensed at the center, since the magnetic effect at the point p of the uniform layer of current in the conductor beyond b will be zero. The strength of field at p is therefore,

$$H_c = \frac{2I}{b} \cdot \frac{\pi b^2}{\pi r^2} = \frac{2Ib}{r^2}$$

and $$B_c = \frac{2\mu' Ib}{r^2}$$

$$D_c\, db = dN_c = \frac{2\mu' Ib\, db}{r^2}$$

$$N_c = \int_0^r \frac{2\mu' Ib\, db}{r^2} = \mu' I$$

But this induction does not link with the whole current but only with that within a circle of radius b. The product of the current with the number of lines of force inclosed by it is

$$\int dN_c\, dI_c = \int_0^r \frac{2\mu' I b\, db}{r^2} \cdot \frac{\pi b^2}{\pi r^2} = \frac{1}{2}\mu' I$$

The self inductance of A per unit of length for unit current is

$$L_A = \left(2\mu \log_e \frac{d}{r} + \frac{\mu'}{2}\right)$$

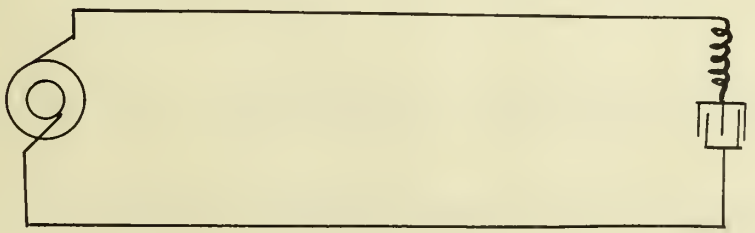
The return conductor doubles the above so therefore the self inductance per unit of circuit is

$$L_{AA'} = 2\left(2\mu \log_e \frac{d}{r} + \frac{\mu'}{2}\right)$$

when the medium is air $\mu = \mu' = 1$

$$\therefore \quad L_{AA'} = 2\left(2 \log_e \frac{d}{r} + \frac{1}{2}\right)$$

CURRENT VOLTAGE AND POWER IN INDUCTIVE LINE OF NEGLIGIBLE CAPACITY



Assume true sine waves.

Let Resistance of line = R

Reactance " " = $\omega L = X$.

Resistance of receiver circuit = R_1

Resistance " " " = $\omega L - \frac{1}{\omega C} = X_1$

Current in " " = I

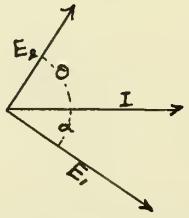
Then the voltage at the receiver circuit is $I\sqrt{R_1^2 + X_1^2} = E_1$, making $\tan^{-1}\frac{X_1}{R_1}$ with I.

The voltage on the line is $I\sqrt{R^2 + X^2} = E$ making $\tan^{-1}\frac{X}{R}$ with I.

Generator voltage is

$$E = E_1 + E_2 = \sqrt{(E_2\cos\theta + E_1\cos\alpha)^2 + (E_1\sin\alpha + E_2\sin\alpha)^2}$$

This is a maximum when $\theta = \alpha$ as it then becomes $E_1 + E_2$.



$E = 0$ if $E_1 = E_2$ and $\alpha = -\theta = -90°$ i.e. if the line and receiver are in resonance and neither contains a resistance which is an impossible case. In any case E is a minimum when $\theta - \alpha$ is a maximum.

If $\alpha > 0$ i.e. if receiving circuit is conductive, then line drop is always a posi-

-tive quantity. It is less for values of α smaller as well as larger than θ, than for the case when $\alpha = \theta$.

If line drop is 0, i.e, if $E = E_1 + E_2 = E_1 =$

$$\sqrt{(E_1 \cos\alpha + E_2 \cos\theta)^2 + (E_1 \sin\alpha + E_2 \sin\theta)^2}$$

then $E_1^2 = E_1^2 + E_2^2 + 2E_1E_2(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

and $\cos(\theta - \alpha) = \frac{-E_2}{2E_1} = \frac{-Z_2}{2Z_1}$

Or $-Z_1^2 = R_1R_2 + X_1X_2$ is the condition for no loss of voltage in line. This can be only when x_1 and x_2 are of opposite sign, i.e. the receiver be a capacity circuit.

When receiver circuit is non-inductive $\alpha = 0$ and

$$E = E_1 + E_2 = \sqrt{(E_1 + E_2\cos\theta)^2 + E_1^2\sin^2\theta}$$

$$= \sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\theta}$$

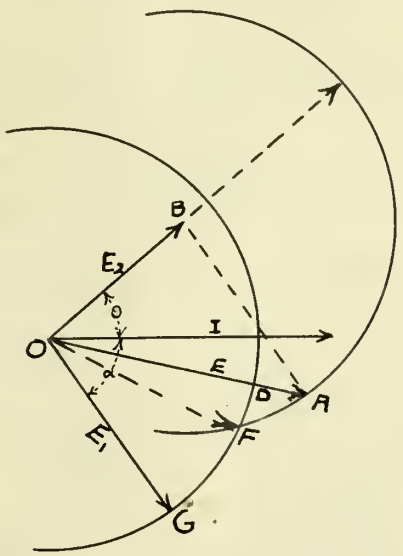
When receiver circuit contains capaci-ty, line drop may be negative. In this case,

$$- \left[2E_1E_2(\sin\theta \sin\alpha + \cos\theta \cos\alpha)\right] > E_2^2$$

In any case line drop

$$= \sqrt{(E_1\cos\alpha + E_2\cos\theta)^2 + (E_1\sin\alpha + E_2\sin\theta)^2} - E_1$$

The above may be shown graphically as follows:-



Let I represent the current; E_2 the voltage caused by current I flowing through the line impedance, making angle θ with I; E_1 the voltage caused by current I flowing through receiver impedance making angle α with I. Then E the vector sum of E_1 and E_2, OB+BA, is the generator E.M.F.

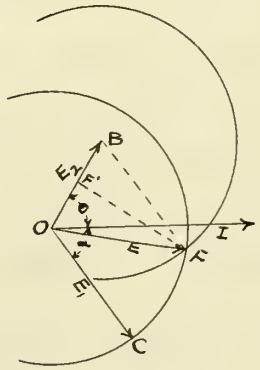
If impedance of receiver circuit is constant but phase angle varies, the point A lies on a circle described about B with $E_1 = OC$ as a radius. line drop $= E - E_1$, ∴ describe about O a circle of radius E_1, intersecting OA in D. Then line drop $= OA - OD = DA$. When $\alpha = \theta$, $E = E_1 + E_2$ and line drop $= E_2$

When line drop $= 0$, $E = OF = E_1$

$$\theta - \alpha = 180° - \angle FBF'$$

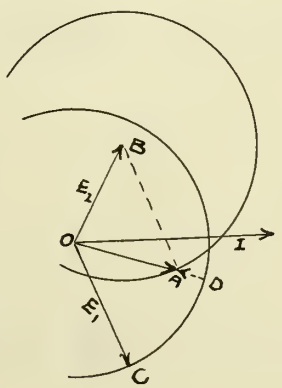
$$\therefore \cos(\theta - \alpha) = -\frac{BF'}{BF} = \frac{-\frac{E_2}{2}}{E_1}$$

$$= \frac{-E_2}{2E_1}$$



as was shown above.

When $-\cos(\theta - \alpha) > +\frac{E_2}{2E_1}$ line drop is negative as shown by direction of line DA, as shown in figure, voltage at end of line being, in this case, higher than at generator.



From the figure it is seen that

$$E = \sqrt{E_1^2 + E_2^2 - 2E_1E_2\cos[180° - (\theta - \alpha)]}$$

$$= \sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos(\theta - \alpha)}$$

as was otherwise derived above.

POWER IN RECEIVER CIRCUIT.

1. NONINDUCTIVE RECEIVER CIRCUIT.

$$P = E_1 I\cos\alpha = (\dot{E}_1 - \dot{E}_2)I$$

$$= \left(\sqrt{E^2 - E_2^2\sin^2\theta} - E_2\cos\theta\right)I$$

$$= \left[\sqrt{E^2 - (IZ_2\sin\theta)^2} - IZ_2\cos\theta\right]I$$

However for line of given resistance more power can be transmitted to non-inductive receiver circuit over non-inductive than over inductive line, as can be seen when above formula is written

$$P = \left[\sqrt{E^2 - I^2Z_2^2\left(\frac{X_2}{Z_2}\right)^2} - IZ_2\frac{R_2}{Z_2}\right]I$$

$$= \left[\sqrt{E - I^2X_2^2} - IR_2\right]I$$

From this it is seen that for a line of

given impedance Z, with given generator e.mf. E, more power is transmitted to a non-inductive receiver circuit over inductive than over noninductive line.

To find the value of current for which maximum power can be transmitted over line having the constants X_2, R_2, equate the first derivative of the expression for power, with respect to I, to o and solve for I.

$$\frac{dP}{dI} = \frac{2E^2I - 4I^3X_2^2}{2\sqrt{E^2I^2 - I^4X_2^2}} - 2IR = 0$$

$$= E^2 - 4I^2Z^2E^2 + 4I^4(X^4 + R^2X^2)$$

$$\therefore \quad I = \sqrt{\sqrt{\frac{-E^2}{4X_2^4 + R_2^2X_2^2} + \frac{E^4}{4X_2^4}} + \frac{E^2}{X_2^2}}$$

$$= \sqrt{\sqrt{-\frac{E^2}{4X_2^2Z_2^2} + \frac{E^2}{X_2^2}} + \frac{E^4}{4X_2^4}}$$

In the general case $P = EI\cos\alpha$

$$E_1^2 + E_2^2 + 2E_1E_2\cos(\theta - \alpha) = E^2$$

$$\therefore \quad E_1 = \sqrt{E^2 - E_2^2 + E_2^2\cos^2(\theta - \alpha)} - E_2\cos(\theta - \alpha)$$

$$\therefore \quad P = \left[\sqrt{E^2 - E_2^2 + E_2^2\cos^2(\theta-\alpha)} - E_2\cos(\theta-\alpha)\right] I\cos\alpha$$

$$= \left[\sqrt{E^2 - I^2Z_2^2\sin^2(\theta-\alpha)} - IZ\cos(\theta-\alpha)\right] I\cos\alpha$$

To find value of I for which P is a maximum θ, α, Z_2 being given equate the first derivative of P with respect to I, to 0 and solve equation for I.

$$\frac{dP}{dI} = \frac{d}{dI}\left[\left[\sqrt{E^2 - I^2Z_2^2\sin^2(\theta-\alpha)} - IZ\cos(\theta-\alpha)\right] I\cos\alpha\right]$$

$$= \left[\frac{1}{2}\frac{2E^2I - 4I^3Z^2\sin^2(\theta-\alpha)}{\sqrt{E^2I^2 - I^4Z^2\sin^2(\theta-\alpha)}} - 2IZ\cos(\theta-\alpha)\right]\cos\alpha = 0$$

$$E^2 - 2I^2Z^2\sin^2(\theta-\alpha) = 2IZ\cos(\theta-\alpha)\sqrt{E^2 - I^2Z^2\sin^2(\theta-\alpha)}$$

$$E^4 - 4E^2I^2Z^2\sin^2(\theta-\alpha) + 4I^4Z^4\sin^4(\theta-\alpha) = 4I^2Z^2\cos^2(\theta-\alpha)\left[E^2 - I^2Z^2\sin^2(\theta-\alpha)\right].$$

$$E^4 - 4E^2I^2Z^2 + 4I^4Z^4\sin^4(\theta-\alpha) = 4I^4Z^4\cos^2(\theta-\alpha)\sin^2(\theta-\alpha)$$

(1) $\quad I^4\left[4Z^4\sin^4(\theta-\alpha) - 4Z^4\cos^2(\theta-\alpha)\sin^2(\theta-\alpha)\right] - 4E^2I^2Z^2 = -E^4$

(2) $\quad \therefore I^2 = \dfrac{\sqrt{4\sin^4(\theta-\alpha) + 4\cos^2(\theta-\alpha)\sin^2(\theta-\alpha)E^4 + \frac{1}{4}E^4Z^4 + \frac{1}{4}E^2Z^2}}{4Z^4\left[\sin^4(\theta-\alpha) + \cos^2(\theta-\alpha)\sin(\theta-\alpha)\right.}$

(3) $\quad \therefore I = \sqrt{\dfrac{-\sqrt{-4\sin^2(\theta-\alpha) + 4\cos^2(\theta-\alpha)\sin^2(\theta-\alpha)E^4 + \frac{E^4Z^4}{4} + \frac{E^2Z^2}{4}}}{4Z^4\left[\sin^4(\theta-\alpha) + \cos^2(\theta-\alpha)\sin^2(\theta-\alpha)\right]}}$

From equation (1) it is seen that for $\alpha = 0$, maximum power is delivered when $I = \frac{E}{2Z}$, a result easily obtained for this case.

With a line of constants X_2, R_2, giving phase angle θ, and a generator voltage E, to transmit power P to receiver circuit of variable phase angle α, a value of α for which I is a minimum can be found by expressing I as a function of α and equating its first derivative with respect to α to 0.

$$P = \left[\sqrt{E^2 - I^2Z^2\sin^2(\theta-\alpha)} - IZ\cos(\theta-\alpha)\right] I\cos\alpha$$

$$E^2 - I^2Z^2\sin^2(\theta-\alpha) = \frac{P^2}{I^2\cos^2\alpha} - \frac{2PIZ\cos(\theta-\alpha)}{I\cos\alpha} + I^2Z^2\cos^2(\theta-\alpha)$$

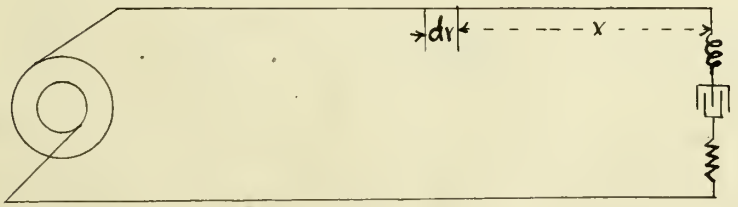
$$(E^2 - I^2Z^2)I^2\cos^2\alpha = P^2 - 2ZPI^2\cos\alpha\cos(\theta-\alpha)$$

$$I = \left[\sqrt{\frac{-P^2}{Z^2\cos^2\alpha} + \frac{1}{4}\left[\frac{E^2\cos^2\alpha + 2PZ\cos\alpha\cos(\theta-\alpha)}{Z^2\cos\alpha}\right]^2} + \frac{E^2\cos^2\alpha + 2PZ\cos\alpha\cos(\theta-\alpha)}{2Z^2\cos^2\alpha}\right]^{\frac{1}{2}}$$

$\frac{dI}{d\alpha}$ equate to 0 and solve for α

For the case of a transmission line having inductance, resistance, leakage and capacity each uniformly distributed, it seems impossible to derive formulae except by use of the complex quantities. This is done by Mr Steinmetz in his work on Alternating Current Phenomena.

For ordinary lines, of 100 miles or less in length, but little error is made in assuming the charging current constant for each element of line.



$i_x = E C x w$ where c is capacity per element, x is distance from end of line.

$d e = E C x w z \, dx$ z is impedance per element

$$\therefore \quad e = \left[\frac{E C w x^2 Z}{2} \right]_0^x = \frac{E C w x}{2} (R_1 + X_1) = \frac{I_c}{2} (R_1 + X_1)$$

Charging current is 90° ahead of E, therefore on open circuit the component $\frac{I_c}{2}R_2$ is 90° ahead of E and the component $\frac{I_c}{2}X$, 180° ahead of E.

∴ On open circuit the voltage at end of line $= \sqrt{(E^2 + \frac{I_c X_2}{2})^2 + \frac{I_c R}{2}}$ behind E by $\tan^{-1}\frac{I_c R}{2E + I_c X_2}$

With receiver circuit of phase angle α current at generator is

$$I_c + I_1 = \frac{E c w}{2} + I$$

if $\alpha = \tan^{-1}\frac{X_2}{R_1}$, $E = I\sqrt{(R_2+R_1)^2 + (X_2+X_1)^2}$

making $\tan^{-1}\frac{X_1+X_2}{R_1+R_2}$ with I.

∴ I_c makes $\tan^{-1} - \frac{R_1+R_2}{X_1+X_2}$ with I, i.e. is ahead of I by this angle. If $X_1 = -X_2$, i.e. if line plus receiver is in resonance $\tan^{-1}\frac{R_1+R_2}{X_1+X_2}$ is $\tan^{-1} - \infty = 90°$. Total current at generator is $\sqrt{I_c^2 + I^2}$. (In this case receiver is a capacity circuit.)

If $X_1 + X_2$ is large, then current at generator is less than if there were no cap-

-acity in the line.

In general total generator current is

$$= I_c + I$$

$$= \sqrt{\left(I + I_c \cos \tan^{-1} \frac{R_1+R_2}{X_1+X_2}\right)^2 + I_c^2 \sin^2 \tan^{-1} \frac{R_0+R_1}{X_0+X_1}}$$

$$= \sqrt{\left(I - I_c \frac{X+X}{\sqrt{(X+X)^2+(R+R)^2}}\right)^2 + I_c^2 \frac{(R_1+R_2)^2}{(R_1+R_2)^2+(X_1+X_2)^2}}$$

From the above it is seen that receiver circuit of a line of large capacity must have low power factor.

To calculate the generator e. m. f. to give a receiver voltage E, it is convenient to divide the receiver current into the components $I \cos \alpha$, in phase with E and the $I \sin \alpha$, $\frac{\pi}{2}$ behind E. Then the e. m. f. at the generator is

$$\sqrt{\left[E + I \cos \alpha R_2 + I \sin \alpha X_2 - \frac{I_c}{2} X_2\right]^2 + \left[I \cos \alpha X_2 - I \sin \alpha R_2 + \frac{I_c}{2} R_2\right]}$$

The most general case is of a trans-
-smission line having distributed capac-

-ity, inductance and leakage. It has al-
-ready been shown that a line has in-
-ductance and capacity. Leakage takes
place over insulators and by direct es-
-cape of electricity into the air. The
leakage current is proportional to and
in phase with the e.m.f..

In considering this case it seems
necessary to use the complex quantity
in order to prevent equations becoming
unwieldly. Let

r = resistance of line per unit length.

x = reactance " " " " " .

g = quantity by which E must be
multiplied to give leakage current per
unit length of line.

b = quantity by which E must be
multiplied to give capacity current per

unit length of line.

Knowing these and the current and e.m.f at any point of the line as at the receiver circuit it is required to find the current and e.m.f. at any other part of the line, as at generator.

Let an element dx, distance x from point o where current is I' and e.m.f. E',

Leakage current $= \dot{E} g\, dx$

Capacity current $= +j \dot{E} b\, dx$ ($+j$ being used to show that current is $\frac{\pi}{2}$ ahead of $\dot{E}$).

$\therefore \quad d\dot{I} = \dot{E}(g + jb)\, dx$

or $\quad \frac{d\dot{I}}{dx} = \dot{E}(g + jb)$ ---------[1].

The e.m.f. in same element due to res. $= I \times dx$

" " " " " " " " " " cap. $= jI \times dx$

or $\quad d\dot{E} = \dot{I}(r + jx)\, dx$

$\therefore \quad \frac{d\dot{E}}{dx} = \dot{I}(r + jx)$ ----------[2]

Differentiate [1] and [2] with respect to x

$$\frac{d^2 I}{dx^2} = \frac{dE}{dx}(g + jb) \quad \text{---------- [3]}$$

$$\frac{d^2 E}{dx^2} = \frac{dI}{dx}(r + jx) \quad \text{---------- [4]}$$

Substitute [1] in [4] and [2] in [3]

$$\frac{d^2 E}{dx^2} = E(g + jb)(r + jx) \quad \text{---------- [5]}$$

$$\frac{d^2 I}{dx^2} = I(g + jb)(r + jx) \quad \text{---------- [6]}$$

These differential equations of E and I are of the same form and are integrated by $w = ae^{vx}$, a being a constant of integration, and v being $\pm\sqrt{(g + jb)(r + jx)}$

$\therefore E = Ae^{vx} + Be^{-vx}$, A and B being constants.

Let $v = \alpha + j\beta$ ---------- [7]

$(\alpha + j\beta)^2 = (g + jb)(r + jx)$

$\therefore \alpha^2 - \beta^2 = gr - bx$

$2\alpha\beta = gx + br$

Substituting $\alpha + j\beta$ for v in $E = Ae^{vx} + Be^{-vx}$

$$E = Ae^{(\alpha + j\beta)x} + Be^{-(\alpha + j\beta)x}$$

$$= Ae^{\alpha x}(\cos\beta x + j\sin\beta x) + Be^{-\alpha x}(\cos\beta x - j\sin\beta x)$$

[BYERLY'S CALCULUS VOL. II P. 22]

$$= (Ae^{\alpha x} + Be^{-\alpha x}) \cos\beta x + j(Ae^{\alpha x} - Be^{-\alpha x}) \sin\beta x$$

$$\frac{dE}{dx} = (\alpha + j\beta)(Ae^{\alpha x} - Be^{-\alpha x}) \cos\beta x + (j\alpha - \beta)(Ae^{\alpha x} + Be^{-\alpha x}) \sin\beta x$$

Substitute this value of $\frac{dE}{dx}$ in equation [3]

$$I(r + jx) = (\alpha + j\beta)\left[(Ae^{\alpha x} - Be^{-\alpha x}) \cos\beta x + j(Ae^{\alpha x} + Be^{-\alpha x}) \sin\beta x\right]$$

$$\text{or } I = \frac{(\alpha + j\beta)}{r + jx}\left[Ae^{\alpha x} - Be^{-\alpha x}) \cos\beta x + j(Ae^{\alpha x} + Be^{-\alpha y}) \sin\beta x\right]$$

At point where $x = 0$

$$I = \frac{\alpha + j\beta}{r + jx}(A + B)$$

$$E = \frac{\alpha + j\beta}{g + jb}(A_E + B_E)$$

To get same values of A, B, in expressions for E and I substitute $\frac{dI}{dx}$ in equation [1]

$$\frac{dI}{dx} = E(g + jb)$$

$$= \frac{\alpha + j\beta}{r + jx}(\alpha + j\beta)\left[(Ae^{\alpha x} + Be^{-\alpha x}) \cos\beta x + j(Ae^{\alpha x} - Be^{-\alpha x}) \sin\beta x\right]$$

$$\therefore \quad E = (Ae^{\alpha x} + Be^{-\alpha x}) \cos\beta x + j(Ae^{\alpha x} - Be^{-\alpha x}) \sin\beta x$$

If $x = 0$, $E = A - B$.

If I and E are known at any point, $x = x_1$, then values of A and B can be found and substituted for values in equations for E and

I, thus giving E and I at any point.

CONCLUSION.

In the development of all the for-
-mulae given above, except the exact formulae for E and I it is assumed that current at all points of the circuit is the same, that the wave forms of current and e. m. f. are true sine waves. That this is not exactly correct has been shown by experiment (Trans. A. I. E. E. april, 1904, by Geo. H. Rowe.) and can be seen by inspection of exact formulae for E and I. The error to which these assumptions lead is not, however, great enough to make these formulae useless for lines no longer than any yet con-
-structed.

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